Optimal paths for minimizing entropy generation in a common class of finite-time heating and cooling processes

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For a common class of finite-time heat transfer processes, we derive optimal heating and cooling strategies for minimizing entropy generation. Solutions pertain to a generalized heat transfer law, and are illustrated quantitatively for cases of practical interest, including Newtonian and radiative heat transfer. Optimal paths are compared with the common strategies of constant heat flux and constant source (reservoir) temperature operation, including evaluation of the savings in entropy generation and the relative requirements for installed heating/cooling capacity.

Keywords: finite-time thermodynamics; entropy generation; optimal control; heat exchange; installed capacity

Introduction

Many energy-intensive heating and cooling processes are constrained to proceed in a fixed given time. Fixed production rates in industry are common examples. For a given energy consumption, different heating and cooling strategies generate different amounts of entropy. Although minimizing entropy generation is usually of little concern to the average individual consumer, it can be of practical value to factories with cogeneration facilities, or to those that reuse process heat at lower temperatures, or to the electric utility providing the energy. Consequently, minimum entropy generation has become an established engineering design criterion (Bejan 1982, 1988). Closely related to this point of view is the availability or second-law analysis (Wepfer 1979; Gaggioli 1980; Moran 1982, 1988).

The present article is not the first to recognize the idea of optimal heating and cooling strategies. Earlier studies considered a model that in some ways is similar to the one analyzed here, except that the reservoir temperature was the single time-dependent temperature of a well-mixed stream of cooling or heating agent, and a linear heat transfer law only was assumed. The sources of irreversibility accounted for were the reservoir-system heat transfer and the dumping of the used stream into the environment.

Bejan (1978) used this model in the context of energy storage to find the optimal duration or charging time that minimizes total entropy generation, subject to the constraint of constant flow rate. Bejan and Schultz (1982) employed this approach to determine the minimum amount of fluid (e.g., cryogen) required during a fixed time period, as well as the associated optimal flow-rate history, temperature history, etc. These methodologies focus on finite-time effects (Andresen et al. 1984) and finite-size systems (Ondrechen et al. 1981) and consist of a mixture of classical thermodynamics, heat and mass transfer, and fluid mechanics (Bejan 1982, 1988).

In this article, we examine heating and cooling strategies that are optimal in the sense of minimizing entropy generation. For specificity and clarity of presentation, we restrict our attention to a simple class of one-node heat transfer problems in which a system of uniform temperature T is heated by an external reservoir of temperature T_0, which is our control variable. In the process of deriving explicit formulas for optimal heating and cooling strategies, we accomplish the following.

(1) We examine the sensitivity of these optimal strategies to the functional dependence on temperature of heat transfer law, taken as proportional to $T_0^n - T^n$. The cases of common practical interest (De Vos 1985; Chen and Yan 1989; Gordon 1990; Yan and Chen 1990) considered here are as follows: $n = +1$ (Newtonian heat conduction); $n = -1$ (heat conduction from linearized irreversible thermodynamics or heat conduction in materials such as metals with a specific temperature-dependent thermal conductivity); and $n = +4$ (radiative heat transfer).

(2) We compare the typical heating and cooling strategies of constant reservoir temperature and constant heat flux against the optimal solution, including total entropy generated and the maximum temperatures required. This also affords an appreciation of the empirical wisdom that has evolved in certain common heating and cooling procedures.

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Received 15 July 1991; accepted 7 January 1992

(3) We prove that only for the specific cases of $n = \pm 1$ is the observation of Salamon et al. (1980) valid that the optimal solution corresponds to a constant rate of entropy generation. For general nonlinear systems, the optimal heating/cooling strategy can differ significantly from that of constant rate of entropy generation (radiative heat transfer being a prominent example) and must be solved numerically from the governing equations.

Illustrative quantitative examples are presented for all cases.

Derivation of optimal heating/cooling strategies

The simple one-node thermal model considered is illustrated in Figure 1. The only nonnegligible thermal resistance is at the heat transfer interface between system and external reservoir, where there is a known thermal conductance κ. In the spirit of optimal control theory and some of our earlier studies (Salamon et al. 1980; Ondrechen et al. 1981; Gordon 1990), the reservoir temperature $T_0(t)$ can be varied at will and without loss as a function of time t in order to heat or cool the system at temperature $T(t)$. (This is the most basic feature that distinguishes the present study from Bejan (1978) and Bejan and Schultz(1982)).

Although this freedom might appear unrealistic to some readers, it actually is not. First, the completely unconstrained $T_0(t)$ that follows from optimal control theory provides the truly optimal path that the hardware designer should attempt to follow. Second, one can realistically tailor the time or position dependence of the reservoir (sometimes referred to as the source stream) temperature. This can be achieved, for example, by appropriate heat-exchanger design (Andresen and Gordon 1992) with possibly uneven allocation of heat-exchanger surface; by variable-current infrared radiative heaters, etc. Finally, in the next two sections, we compare this optimal temperature path with the more standard constant reservoir temperature and constant heat-flux strategies. For specificity we present the derivations for heating only ($T_0 > T$), with the solutions for cooling involving a simple change of sign.

The rate of heat transfer q is taken to have the general form

$$q = \kappa_n(T_0^n - T^n) \tag{1}$$

with a generalized thermal conductance κ_n. For consistency of notation, κ_n is negative for negative n. Entropy generation S^u occurs at a rate

$$\frac{dS^u}{dt} = q\left(\frac{1}{T} - \frac{1}{T_0}\right) = \kappa_n(T_0^n - T^n)\left(\frac{1}{T} - \frac{1}{T_0}\right) \tag{2}$$

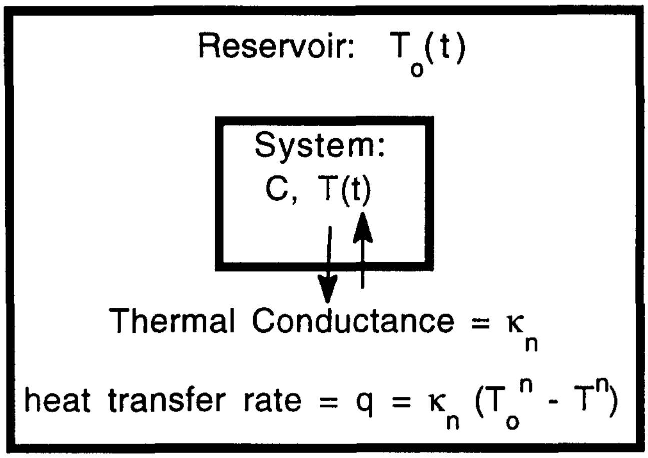

Figure 1 Schematic of one-node thermal model for a system exchanging heat with a variable-temperature (controllable) reservoir

which is always nonnegative as required by the second law of thermodynamics. The change of system temperature $T(t)$ is governed by

$$q = \kappa_n(T_0^n - T^n) = C\frac{dT}{dt} \tag{3}$$

where C is the system's heat capacity.

In a fixed time τ, the system must be heated from a known initial temperature $T(0)$ to a known final temperature $T(\tau)$. Our objective is to minimize S^u subject to the constraint of Equation 3. One proceeds by defining the modified Lagrangian L with a time-dependent Lagrange multiplier $\lambda(t)$:

$$L = \kappa_n(T_0^n - T^n)\left(\frac{1}{T} - \frac{1}{T_0}\right) - \lambda(t)\left\{\kappa_n(T_0^n - T^n) - C\frac{dT}{dt}\right\} \tag{4}$$

The independent variables are T, dT/dt, and T_0 (and in principle dT_0/dt, which, however, does not appear in L). The Euler–Lagrange equations to determine the optimal strategy are then

$$\frac{\partial L}{\partial T} - \frac{d}{dt}\frac{\partial L}{\partial(dT/dt)} = 0 \tag{5}$$

$$\frac{\partial L}{\partial T_0} = 0 \tag{6}$$

For arbitrary n, there is no closed-form, analytic solution;

Notation

C	Heat capacity of the system being heated/cooled
L	Lagrangian function
n	Exponent in the governing heat transfer law
q	Heat transfer rate
S^u	Entropy generation of the universe
S^u_{opt}	Entropy generation of the universe using the optimal strategy
t	Time
T	Temperature of system being heated/cooled
T_0	External reservoir temperature, treated as the control variable
ΔT	Temperature difference through which the system is heated

Greek symbols

α	Constant in general equations for optimal temperature schedules
β	Constant in equations for optimal temperature schedules for $n = 1$
γ	Constant in equations for optimal temperature schedules for $n = -1$
κ_n	Generalized thermal conductance for the T^n heat transfer law
λ	Lagrange multiplier
τ	Process duration

rather it must be generated numerically. Equations 3 through 6 can be reduced to two coupled equations in $T(t)$ and $T_0(t)$, for example:

$$T_0^n - T^n = \alpha T_0^{(n+1)/2} \tag{7}$$

$$\frac{dT_0}{dt} = \frac{\frac{n\kappa_n\alpha}{C} T_0^{(n+1)/2}\{T_0^n - \alpha T_0^{(n+1)/2}\}^{(n-1)/n}}{nT_0^{n-1} - \frac{\alpha(n+1)}{2} T_0^{(n-1)/2}} \tag{8}$$

with a constant α that is determined from a knowledge of $T(0)$ and $T(\tau)$. With the solutions to Equations 7 and 8, S^u is obtained by numerically integrating Equation 2:

$$S^u = \kappa_n \int_0^\tau (T_0^n - T^n)\left\{\frac{1}{T} - \frac{1}{T_0}\right\} dt. \tag{9}$$

We proceed by analyzing these solutions for three cases of practical interest—$n = +1$, $n = -1$, and $n = 4$—and comparing the conventional heating strategies of constant reservoir temperature and constant heat flux against the optimal solutions.

Results for $n = +1$

Optimal solution

For $n = +1$, the governing equations in the previous section can be solved in closed form, yielding

$$T_0(t) = \beta T(t) \tag{10}$$

$$T(t) = T(0)\exp\left\{\frac{\kappa_1(\beta - 1)t}{C}\right\} \tag{11}$$

with β being a constant that is determined from a knowledge of $T(0)$ and $T(\tau)$:

$$\beta = 1 + \frac{C}{\kappa_1\tau}\ln\frac{T(\tau)}{T(0)} \tag{12}$$

The reservoir temperature (control variable) is then implemented according to Equations 10 to 12. The entropy generation using this optimal strategy S_{opt}^u is obtained by integrating Equation 9 with the solutions of Equations 10 to 12, yielding

$$S_{opt}^u = \frac{\kappa_1(\beta - 1)^2\tau}{\beta} \tag{13}$$

(The equilibrium solution of $\beta = 1$ or $T_0(t) = T(t)$ yields zero entropy generation but cannot satisfy the constraint that the process be completed in a fixed given time τ.)

One can now compare ordinarily implemented heating strategies against the optimal solution in terms of (1) the time-dependent reservoir temperature $T_0(t)$; (2) the entropy-generated S^u; and (3) the maximum required reservoir temperature T_0^{max} (equivalently, installed capacity). For example, it has been noted that in single-pass heat exchangers, the commonly preferred choice of counterflow design happens to become identical to the optimal solution for judiciously chosen heat-exchanger parameters (Andresen and Gordon 1992). Here we consider the general cases of constant reservoir temperature and constant heat-flux strategies.

Fixed reservoir temperature

For heating at fixed reservoir temperature, Equation 3 and the boundary conditions require that

$$T(t) = T_0 - [T_0 - T(0)]\exp(-\kappa_1\tau/C) \tag{14}$$

$$T_0 = \frac{T(\tau) - T(0)\exp(-\kappa_1\tau/C)}{1 - \exp(-\kappa_1\tau/C)} \tag{15}$$

Defining the given temperature difference as $\Delta T = T(\tau) - T(0)$, we can express the entropy generation for the constant reservoir temperature strategy as

$$S^u = C\left(\ln\left\{\frac{T(\tau)}{T(0)}\right\} - \frac{\Delta T}{T_0}\right) \tag{16}$$

As can easily be verified from the governing equations (Equations 3 and 9), Equation 16 is valid for all values of n, and not just for $n = 1$.

Constant heat flux

For constant heat flux, the corresponding solutions are

$$T(t) = T(0) + \frac{\Delta T t}{\tau} \tag{17}$$

$$T_0(t) = T(0) + \frac{C\Delta T}{\kappa_1\tau} + \frac{\Delta T t}{\tau} \tag{18}$$

$$S^u = C\ln\left\{\frac{T(\tau)}{T(0)}\right\} - C\ln\left\{\frac{T(\tau) + \frac{C\Delta T}{\kappa_1\tau}}{T(0) + \frac{C\Delta T}{\kappa_1\tau}}\right\} = C\ln\left\{\frac{T(\tau)}{T(0)}\right\} - C\ln\left\{\frac{T_0(\tau)}{T_0(0)}\right\} \tag{19}$$

Overall comparisons

As a quantitative illustration of the relative merits of these alternative heating methods, we consider a low-temperature ($\Delta T = 100$ K) and high-temperature ($\Delta T = 600$ K) heating process starting at $T(0) = 300$ K. The three heating strategies are plotted in Figures 2a and 2b. Relative savings in entropy generation are summarized in Table 1. Constant heat flux is always superior to the constant reservoir temperature procedure, which can be verified by comparing Equation 16 (by using Equation 15 in Equation 16) and Equation 19.

The differences among the different heating strategies become more pronounced as $\kappa_1\tau/C$ (the ratio of process time to system relaxation time) is increased. For example, in the long-time limit of $\kappa_1\tau/C \gg 1$, the entropy generation for the constant reservoir temperature procedure remains finite, while the entropy generation for the optimal and constant heat-flux techniques both vanish, even though at a finite ratio. This limit then affords an upper bound on the superiority of the optimal heating strategy. For the lower-temperature process considered here, the optimal strategy is only 1% superior (in entropy-generation savings) compared to constant heat-flux operation in the long-time limit, whereas both are infinitely better than constant reservoir temperature heating. For the higher-temperature process, the corresponding figure is 11%.

The optimal strategy is a continuous, monotonic function that would appear to be easy to implement. The key pragmatic difficulty is the need to accommodate a larger range of reservoir temperatures than for the common, simpler heating strategies.

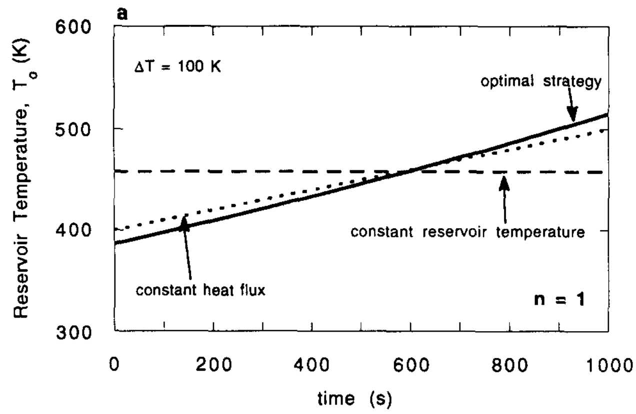

Figure 2(a) Reservoir temperature T_0 *versus* time for heating the system with $C/\kappa_1 = 1000$ s from 300 K to 400 K, i.e., $\Delta T = 100$ K, for the three strategies of (1) constant reservoir temperature T_0; (2) constant heat flux q; and (3) optimal path. The heat transfer law is $q = \kappa_1(T_0 - T)$, i.e., $n = 1$

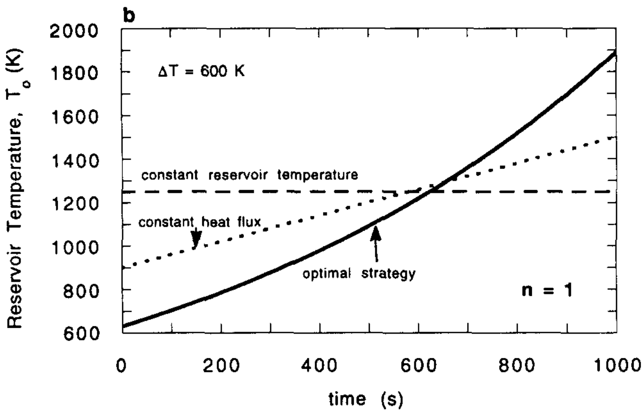

Figure 2(b) As in Figure 2(a), but for heating the system from 300 K to 900 K, i.e., $\Delta T = 600$ K

Table 1 Relative savings in entropy generation[a]

n	ΔT (K)	$S^u_{\text{constant-}q}/S^u_{\text{optimal}}$	$S^u_{\text{constant-}T_0}/S^u_{\text{optimal}}$
1	100	1.00	1.08
1	600	1.02	1.08
−1	100	1.00 (exact)	1.20
−1	600	1.00 (exact)	1.55
4	100	1.00	1.02
4	600	1.01	1.05

[a]Comparison among three heating strategies: (1) optimal; (2) constant heat flux q; and (3) constant reservoir temperature T_0. See captions of Figures 2–4 for system parameters.

Comparison of the three strategies considered shows that paths that generate more entropy require smaller installed heating capacities. As will be shown in the next two sections, these latter observations also pertain to the cases of $n = -1$ and $n = 4$.

Results for $n = -1$

For $n = -1$, the governing equations of the second section of this article yield closed-form solutions for the *optimal heating strategy*:

$$\frac{1}{T} - \frac{1}{T_0} = \gamma \tag{20}$$

$$T(t) = T(0) + \frac{|\kappa_{-1}|\gamma t}{C} \tag{21}$$

(recall $\kappa_{-1} < 0$ by definition), where γ is a constant determined from a knowledge of ΔT:

$$\gamma = \frac{C\Delta T}{|\kappa_{-1}|\tau}. \tag{22}$$

One immediately sees from Equation 20 that the optimal solution is the same as that of *constant heat flux*. For the optimal solution, Equations 20 to 22 applied to Equation 9 yield the entropy produced:

$$S^u_{\text{opt}} = \frac{C^2\Delta T^2}{\tau|\kappa_{-1}|} \tag{23}$$

For operation at *constant reservoir temperature*, the value of T_0 can be determined from Equation 3 to be the solution to the following equation:

$$T_0^2 \ln\left\{\frac{T_0 - T(\tau)}{T_0 - T(0)}\right\} + T_0\Delta T = -\frac{|\kappa_{-1}|\tau}{C} \tag{24}$$

As noted in the previous section, the entropy generation in this case is independent of n and is given by Equation 16.

Comparative plots are presented in Figures 3a and 3b. In order to generate heat transfer rates that are comparable to those of the $n = +1$ case, one must change $C/|\kappa_{-1}|$ with $T(\tau)$. Accordingly, a markedly reduced value of $C/|\kappa_{-1}|$ is selected for the high-temperature process, relative to that for the low-temperature process. Table 1 presents the relative savings in entropy generation.

Results for $n = 4$

Unlike the cases of $n = \pm 1$, closed-form solutions do not emerge for the case of $n = 4$ (radiative heat transfer). One must solve Equations 7 and 8 numerically for the *optimal heating strategy* and then integrate Equation 9 numerically to obtain the minimal possible entropy generation.

At *constant heat flux*, the solutions for $T(t)$ and $T_0(t)$ follow from Equation 3:

$$T(t) = T(0) + \frac{\Delta T t}{\tau} \tag{25}$$

$$T_0(t) = \left\{\left(T(0) + \frac{\Delta T t}{\tau}\right)^4 + \frac{C}{\kappa_4}\frac{\Delta T}{\tau}\right\}^{1/4} \tag{26}$$

Equation 9 is then integrated numerically to obtain S^u.

At *constant reservoir temperature*, the value of T_0 is obtained by solving Equation 3, which affords T_0 as the solution to the following equation:

$$\frac{4\kappa_4 T_0^3 \tau}{C} = \ln\left[\frac{\{T_0 + T(\tau)\}\{T_0 - T(0)\}}{\{T_0 + T(0)\}\{T_0 - T(\tau)\}}\right] + 2\tan^{-1}\left\{\frac{T(\tau)}{T_0}\right\} - 2\tan^{-1}\left\{\frac{T(0)}{T_0}\right\} \tag{27}$$

Sample results for the low- and high-temperature heating processes are shown in Figures 4a and 4b. In order to generate heat transfer rates that are comparable to those of the $n = 1$ case, a reduced value of C/κ_4 is selected for the high-

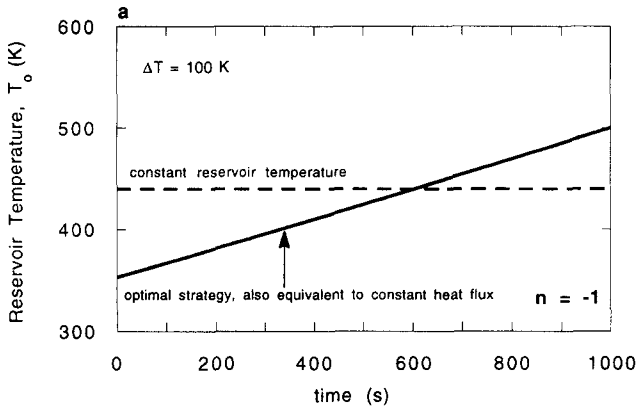

Figure 3(a) As in Figure 2(a), but with the heat transfer law $q = \kappa_{-1}(T_0^{-1} - T^{-1})$, i.e., $n = -1$. $C/|\kappa_{-1}| = 0.005\ K^{-2}\ s$

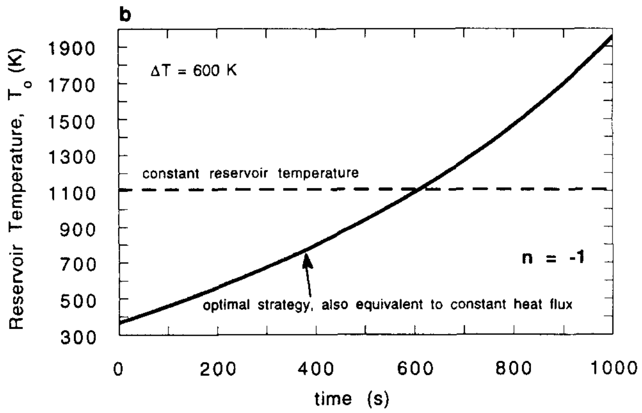

Figure 3(b) As in Figure 3(a), but with $\Delta T = 600$ K. $C/|\kappa_{-1}| = 0.001\ K^{-2}\ s$

temperature process relative to that for the low-temperature process. As evidenced by Table 1, differences in entropy generation among the three strategies considered here are quite small and are less pronounced than for the other heat transfer laws.

Optimal strategy *versus* constant rate of entropy generation

Salamon et al. (1980) proved that for any linear finite-time process, the strategy that minimizes entropy generation is the one that corresponds to a constant rate of entropy generation. This proof pertains directly to the $n = +1$ case. Indeed, substitution of the optimal solution (Equation 10) into Equation 2 confirms that it is equivalent to a constant rate of entropy generation. Is this observation valid for nonlinear heat transfer? One can prove that it is not.

The optimal strategy obeys Equation 7:

$$T = \{T_0^n - \alpha T_0^{(n+1)/2}\}^{1/n} \tag{28}$$

where α is a constant. Substituting Equation 28 into the expression for the rate of entropy generation (Equation 2), one obtains

$$\frac{dS^u}{dt} = \kappa_n \alpha T_0^{(n-1)/2}\{-1 + [1 - \alpha T_0^{(1-n)/2}]^{-1/n}\} \tag{29}$$

which is *not* constant *except* for $n = \pm 1$. Hence in general, for non-linear problems, a constant rate of entropy generation is not the optimal strategy.

Summary

Many energy-intensive heating and cooling procedures are carried out either at constant heat flux or by using a constant temperature source (reservoir). In industrial and electric-utility settings, these processes are often constrained to proceed in a given fixed time (fixed production rates). Although energy consumption may be fixed, different heating and cooling strategies generate differing amounts of entropy.

We have derived the optimal heating/cooling strategy that minimizes entropy generation for a common type of finite-time heat transfer process, approximated by a one-node thermal model. The objective and methodology of minimizing entropy production in energy-intensive processes are not new (Bejan 1978; Wepfer 1979; Salamon et al. 1980; Gaggioli 1980; Ondrechen et al. 1981; Moran 1982; Bejan 1982; Bejan and Schultz 1982; Andresen et al. 1984; De Vos 1985; Moran and Shapiro 1988; Bejan 1988; Chen and Yan 1989; Gordon 1990; Yan and Chen 1990; Andresen and Gordon 1992), and neither is the aim of developing optimal heating and cooling procedures. However, we add a new twist to the optimization by (1) allowing a controllable time-varying reservoir; (2)

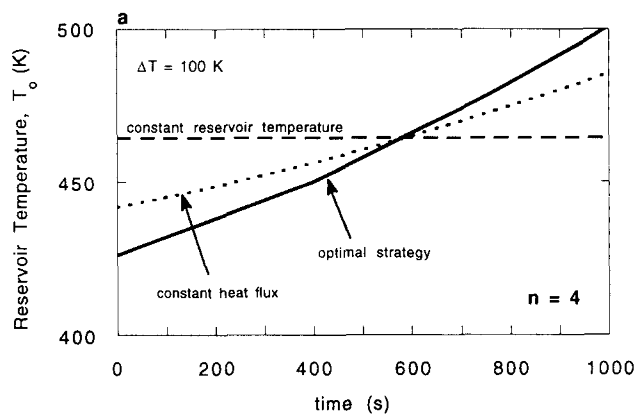

Figure 4(a) As in Figure 2(a), but with the heat transfer law $q = \kappa_4(T_0^4 - T^4)$, i.e., $n = 4$. $C/\kappa_4 = 3 \times 10^{11}\ K^3\ s$

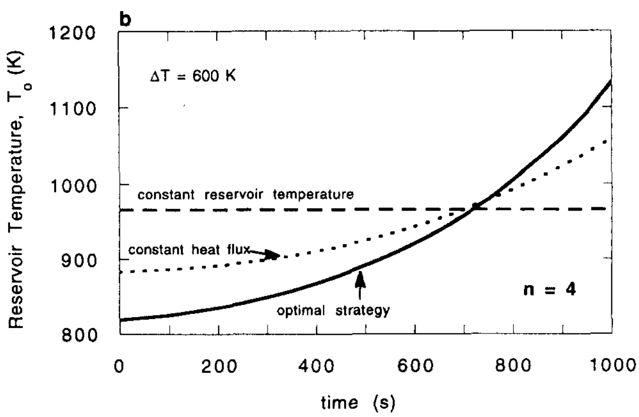

Figure 4(b) As in Figure 4(a), but with $\Delta T = 600$ K. $C/\kappa_4 = 1 \times 10^{12}\ K^3\ s$

considering the sensitivity to heat transfer law; and (3) quantifying the superiority of optimal heating and cooling strategies relative to conventional procedures.

Our derivation pertains to a generalized heat transfer law where heat flux is proportional to $T_0^n - T^n$ (T_0 = external reservoir temperature, which is the time-dependent control variable, and T = system temperature). Cases of practical interest noted to date include $n = +1$, $n = -1$, and $n = 4$ (De Vos 1985; Chen and Yan 1989; Gordon 1990; Yan and Chen 1990), for all of which we have presented quantitative illustrative examples.

It had been previously proposed that optimal paths are those that generate entropy at a constant rate (Salamon et al. 1980). The original proof was developed for linear systems only. We have proven that this theorem is correct only for the cases $n = \pm 1$ with significant deviations for nonlinear heat transfer such as radiation.

The potential savings in entropy generation achievable with the optimal strategy increase as the ratio of process time to system relaxation time increases. These savings also become more pronounced as heat transfer grows progressively nonlinear and as temperature gradients increase.

Operating at a fixed reservoir temperature turns out to be markedly inferior to the optimal strategy as well as to constant heat-flux operation. For the kinds of illustrative examples considered here, working at constant heat flux is typically only 1–10% inferior to the optimal strategy.

The optimal solutions for $T_0(t)$ are continuous monotonic functions that should be easy to implement. Their major practical drawback is that they require a larger installed heating/cooling capacity, in that a larger than normal range of reservoir temperatures is required. The magnitude of this incremental need can be seen in Figures 2a to 4b. Considering the three strategies of constant reservoir temperature, constant heat flux, and optimal operation, one sees that the paths that generate more entropy benefit from necessitating small installed capacities. Hence, the eventual selection of economically optimal strategies will depend on the relative costs of avoided entropy generation *versus* installed capacity. The solutions presented here provide a quantitative basis for such evaluations.

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